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HOMOCLINIC ORBITS IN A NEAR-INTEGRABLE MIXED TYPE CNLS SYSTEM

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We consider a system of coupled nonlinear Schrödinger equations with even, periodic boundary conditions, which are damped and quasi-periodically forced. Under certain conditions, we establish criteria for the existence of homoclinic orbits to a spatially independent invariant torus. We compare the analysis with rigorous numerical simulation.

1. Introduction

Soliton pulses are the heard of high speed fiber-optic telecommunication system and hold great potential for all optical switching devices. Novel soliton packing schemes propose to subdivide information streams into different wavelengths-wavelength division multiplexing (WDM), or into orthogonal polarizations-polarization division multiplexing (PDM). However, a proper modeling of these schemes, especially to address stability properties, requires careful attention to the various perturbations which are present in optical systems. To model the interaction of the orthogonal polarizations of a pulse, the widely studied nonlinear Schrödinger equation (NLS) is extended to the integrable Manakov system. However, fiber-optic systems exhibit birefringence, differing phase and group velocities for different polarizations, as well nonlinear interactions between polarizations dependent upon amplitude-cross-phase modulation (XPM) and upon complex phase-four-wave mixing (FWM). The nonlinear terms break the integrability of the Manakov equation and numerical simulations including these nonlinearities have demonstrated pulse splitting and inelastic collisions.

We consider the mixed type coupled NLS equations:

$$\begin{aligned} ip_t &= -p_{xx} - \frac{1}{2}(\sigma_1 |p|^2 + \sigma_2 |q|^2 - \omega^2)p \\ &\quad + i\varepsilon [a_1 p + d_1 p_{xx} - \Gamma_1 - \nu p_x + (|p|^2 + \delta_1 |q|^2)], \\ iq_t &= -q_{xx} - \frac{1}{2}(\sigma_1 |p|^2 + \sigma_2 |q|^2 - \omega^2)q \end{aligned} \tag{1}$$

$$+ i\varepsilon [a_2 q + d_2 q_{xx} - \Gamma_2 - \nu q_x + (\delta_1 |p|^2 + |q|^2)],$$

with even periodic boundary conditions

$$p(x + 2\pi) = p(x), \quad q(x + 2\pi) = q(x), \quad p(-x) = p(x), \quad q(-x) = q(x), \quad (2)$$

where $a_1, a_2, d_1, d_2, \delta_1, \Gamma_1, \Gamma_2, \nu$ are small parameters and $\delta_1 \neq 1$.

The system (1) is an example of an infinite dimensional dynamical system in interaction. We explicit in the next section its corresponding symplectic structure. Previous examples of such structures have been presented for the coupled Maxwell-Dirac system⁹ and by Chernof and Marsden⁵.

In Section 2, we exhibit the analytic expression of the homoclinic solutions for CNLS and we prove the existence of a set of invariant in order to derive the Mel'nikov conditions to establish necessary condition for the persistence of homoclinic solutions under dissipative perturbations. The sufficient conditions and full dynamical systems consideration are studied in Aigner^{1,2} *et al.* In Section 3, we investigate numerically the existence of temporal homoclinic chaos for the CNLS system.

2. Integrable Homoclinic Orbits

We consider the CNLS system in the following form:

$$ip_t + p_{xx} + \frac{1}{2}p(pr - \omega^2 + qs) = 0, \quad iq_t + q_{xx} + \frac{1}{2}q(pr + qs - \omega^2) = 0, \quad (3)$$

where $r = \sigma_1 p^*$ and $s = \sigma_2 q^*$ under the assumption that the potential are even periodic functions on $[0, L]$.

This system is the compatibility condition of the Lax pair:

$$\vec{\psi}_x = (E\mathbf{A}_0 + \mathbf{A}_1)\vec{\psi}, \quad \vec{\psi}_t = (E^2\mathbf{A}_0 + E\mathbf{A}_1 + \mathbf{A}_2)\vec{\psi}, \quad (4)$$

where $\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2$ are defined in Forest⁷ *et al.* Now consider the plane wave solution

$$p_0 = ae^{ikx - i\Omega t}, \quad q_0 = be^{-ikx - i\Omega t}, \quad (5)$$

$$\text{where} \quad \Omega = k^2 - \frac{1}{2}(\sigma_1 |a|^2 + \sigma_2 |b|^2) + \omega^2.$$

A spatial period is chosen, for example $L = 2\pi$. And then the wavenumber of the plane wave is chosen, for example $k = 1$, and the amplitudes are fixed, for example $a = 2$ and $b = 2$. The plane wave may be linearly unstable to perturbation by certain Fourier modes on the given spatial period. Let the wavenumbers of the Fourier modes be labeled by $\Delta_n = \frac{2\pi}{L}n$.

To construct the homoclinic orbit, we need a certain spectral parameter E_n that corresponds to the wavenumber Δ_n of the Fourier mode. The

plane wave is unstable if and only if E_n is non-real. Assuming $E = E_n$ is non-real, the homoclinic orbit p_h, q_h , which saturates the linear instability, is defined by

$$p_h = p_0 + 2i(E^* - E) \frac{u^*}{\sigma_1 + |u|^2 + |v|^2}, \quad q_h = q_0 + 2i(E^* - E) \frac{v^*}{\sigma_2 + |u|^2 + |v|^2},$$

where $*$ denotes complex conjugate. The variables u and v are defined by in terms of the eigenvector of the Lax pair $\vec{\psi} = (\psi_1, \psi_2, \psi_3)^t$ by $u = \psi_2/\psi_1, v = \psi_3/\psi_1$. Upon substitution into the Bäcklund transformation⁷, and $\sigma_1 a^2 + \sigma_2 b^2 = \lambda^2 + \Delta^2$, so $\theta_0 \in [0, \pi/2]$ as defined above makes sense. The expression of the homoclinic solution for the CNLS system (since $k = 0$) is given by:

$$p_h = p_0 h(x, t), \quad q_h = q_0 h(x, t), \quad (6)$$

where

$$h(x, t) = \frac{\cos 2\theta_0 - \sin \theta_0 \operatorname{sech} 2\tau \cos(2\chi + \theta_0 - \frac{\pi}{2}) + i \tanh 2\tau \sin 2\theta_0}{1 + \sin \theta_0 \operatorname{sech} 2\tau \cos(2\chi + \theta_0 - \frac{\pi}{2})}.$$

The orbit is homoclinic *up to a phase shift* to the base plane wave.

$$\lim_{t \rightarrow \pm\infty} h(x, t) = e^{\pm 2\theta_0 i}, \quad \theta_0 = \arctan\left(\frac{\lambda}{\Delta}\right), \quad \lambda^2 = \sigma_1 a^2 + \sigma_2 b^2 - \Delta^2. \quad (7)$$

The Floquet discriminant \mathbf{D} is central to the theory of CNLS. We consider the initial value problem:

$$\mathbf{L}\mathbf{M} = E\mathbf{M}, \quad \mathbf{M}(0; E; p, q, r, s) = \mathbf{I}, \quad (8)$$

with

$$\mathbf{L} = \partial_x - (EA_0 + \mathbf{A}_1), \quad (9)$$

where A_0, \mathbf{A}_1 are defined in (4). The Floquet Discriminant \mathbf{D} is defined by

$$\mathbf{D} = -4(\mathbf{D}_1^3 + \mathbf{D}_2^3) + \mathbf{D}_1^2 \mathbf{D}_2^2 + 18\mathbf{D}_1 \mathbf{D}_2 - 27, \quad (10)$$

$$\mathbf{D}_1 = \operatorname{trace} \mathbf{M}(l) = M_{11}(l) + M_{22}(l) + M_{33}(l),$$

$$\mathbf{D}_2 = M_{11}(l)M_{22}(l) - M_{12}(l)M_{21}(l) + M_{22}(l)M_{33}(l) - M_{23}(l)M_{32}(l) \\ + M_{11}(l)M_{33}(l) - M_{13}(l)M_{31}(l).$$

One interplays \mathbf{D} 's dependence upon the complex spectral parameter E with its dependence of the functions p, q and r, s . The function \mathbf{D} is entire in both E and (p, q, r, s) and we have used its E dependence to characterize the spectrum of the operator \mathbf{L} .

If the potentials are sufficiently smooth, one can describe the asymptotic behavior of $\mathbf{D}(E; p, q, r, s)$ as $E \rightarrow \infty$. The gradients of $\mathbf{D}_1, \mathbf{D}_2$ and the Floquet Discriminant \mathbf{D} with respect to (p, q, r, s) are given in Aigner¹ *et al.*

3. Persistence Homoclinic Solutions-Necessary Condition

In this section, we discuss the derivation of transversality condition based on the Poincaré-Arnold-Melnikov theory for finite dimensional dynamical systems and the Hamiltonian structure of the CNLS system. The evolution of any real-valued functional \mathcal{S} under the flow governed by the unperturbed CNLS formally obeys:

$$\frac{d\mathcal{S}}{dt} = \{ \mathcal{S}, H_0 \}.$$

The unperturbed CNLS flow conservates the following quantities: (i) the momentum of the solutions J_1 (ii) the energy of the first mode J_2 and the second mode J_3 (iii) the Floquet discriminant D , Aigner¹ *et al.* The above functionals conserved by the CNLS flow are those that Poisson commute with the Hamiltonian

$$\begin{aligned} H_0(p, q, \bar{p}, \bar{q}) &= \frac{1}{2\pi} \int_0^{2\pi} (|p_x|^2 + |q_x|^2 + \frac{1}{2}|p|^2|q|^2) + \frac{1}{4}(\sigma_1|p|^4 + \sigma_2|q|^4) \\ &\quad + \frac{\omega^2}{2}(|p|^2 + |q|^2) dx, \\ \{J_1, H_0\} &= \{J_2, H_0\} = \{J_3, H_0\} = \{D, H_0\} = 0. \end{aligned} \quad (11)$$

The class of functional are related with the special symmetries of solutions of CNLS. In particular, we consider the complex functions $p(x, t), q(x, t)$ as $p = Pe^{ia}, q = Qe^{ib}$ and using the Lie symmetries theory one can prove that the solutions p, q of CNLS are invariant by space translation, time translation, rotation of the phase a, rotation of the phase b. Hence, Noether's theorem implies that there exist four independent nontrivial conserved quantities associated with each of the Hamiltonian symmetries (J_1, H_0, J_2, J_3) . Using the Hamiltonian structure of the CNLS system and implicit function theorem, we can prove¹:

Prop 3.1. A solution $\mathbf{q} = (p, q)$ of CNLS which is homoclinic to a two dimensional torus of fixed points persists under the perturbation (cf (1) if there exists \mathbf{q}_ε ε -dependent family of solutions of the PCNLS such that:

$$\lim_{|t| \rightarrow \infty} \mathcal{S}(\mathbf{q}_\varepsilon(t)) = \mathcal{S}(Q_\varepsilon), \quad \lim_{|t| \rightarrow \infty} \Phi_\mathcal{S}(\mathbf{q}_\varepsilon(t)) = \Phi_\mathcal{S}(Q_\varepsilon) = 0, \quad \Phi_\mathcal{S}(\mathbf{q}_\varepsilon(t)) \equiv \Phi_\mathcal{S}(Q_\varepsilon),$$

as $\varepsilon \rightarrow 0$ and Q_ε is a perturbed saddle point. For any functional \mathcal{S} that Poisson commutes with H_0 , we define the Melnikov function as:

$$M = \int_{-\infty}^{+\infty} \Phi_\mathcal{S}(\mathbf{q}_{\text{hom}}) dt,$$

$$\begin{aligned}
&= \int_{-\infty}^{+\infty} \left[\left\{ \mathcal{S}, H_1 \right\} + \int_0^{2\pi} \frac{\partial \mathcal{S}}{\partial \bar{p}} \left(\frac{\partial F}{\partial p} - \nu p_x \right) + \frac{\partial \mathcal{S}}{\partial \bar{q}} \left(\frac{\partial F}{\partial q} - \nu q_x \right) \right. \\
&\quad \left. + \frac{\partial \mathcal{S}}{\partial p} \left(\frac{\partial F}{\partial \bar{p}} - \nu \bar{p}_x \right) + \frac{\partial \mathcal{S}}{\partial q} \left(\frac{\partial F}{\partial \bar{q}} - \nu \bar{q}_x \right) dx \right] dt, \tag{12}
\end{aligned}$$

where

$$\begin{aligned}
H_1(p, q, \bar{p}, \bar{q}) &= \frac{i}{2\pi} \int_0^{2\pi} \Gamma_1(p - \bar{p}) + \Gamma_2(q - \bar{q}) dx, \\
F(p, q, \bar{p}, \bar{q}) &= \frac{1}{2\pi} \int_0^{2\pi} d_1 |p_x|^2 + d_2 |q_x|^2 - (a_1 |p|^2 + a_2 |q|^2) \\
&\quad + \frac{1}{2} (|p|^4 + |q|^2 + \delta |p|^2 |q|^2) dx.
\end{aligned}$$

Suppose that there exists a point $(\theta_1^\pm, \theta_2^\pm, \hat{\delta}_0)$ such that

$$M(\theta_0, \hat{\delta}_0) = 0 \quad \text{and} \quad \partial_{\theta_0} M(\theta, \hat{\delta}_0) \neq 0,$$

then the Mel'nikov function has simple zeros, ie $W^u(\vec{q}_\varepsilon)$ and $W^s(\mathcal{M}_\varepsilon)$ intersect transversally.

4. Numerical Simulation

The equations given by (1.1) are integrated using a pseudospectral method using a Fourier cosine series satisfying the even boundary conditions given by (1.2). For the temporal integration a fifth order variable-stepsizes Runge-Kutta method is used with local error control. A spatial discretization of $nx = 128$ and a stepsize of about $dt = 10^{-5}s$ works in all cases. A fast discrete cosine differentiation matrix is used to improve performance.

Theoretically homoclinic solutions exist only for the focusing-focusing case ($\sigma_1 = \sigma_2 = 1$) and the mixed focusing-defocusing case ($\sigma_1 = \pm 1, \sigma_2 = \mp 1$). To find the homoclinic solutions to the perturbed CNLS system we initialize with a complex perturbation \tilde{p}, \tilde{q} to a plane wave solution

$$p = p_0(1 + \delta \tilde{p}), \quad q = q_0(1 + \delta \tilde{q}), \tag{13}$$

where the plane wave solution of constant amplitudes a, b are given by equation (2.3). The complex perturbations \tilde{p}, \tilde{q} can be expressed as linear combinations of pure Fourier modes (see Forest⁷ *et al* and Forest and Wright⁸) given by

$$\tilde{p} = f_+ e^{i(\kappa x - \mu t)} + f_-^* e^{-i(\kappa x - \mu^* t)}, \quad \tilde{q} = g_+ e^{i(\kappa x - \mu t)} + g_-^* e^{-i(\kappa x - \mu^* t)}, \tag{14}$$

where $*$ denotes complex conjugate and the coefficients $f_+, f_-, g_+, g_- \in \mathbb{C}$ are given by the eigenvector

$$(f_+, f_-, g_+, g_-) = (+\mu - \kappa)^{-1}, (-\mu - \kappa)^{-1}, (+\mu - \kappa)^{-1}, (-\mu - \kappa)^{-1}, \tag{15}$$

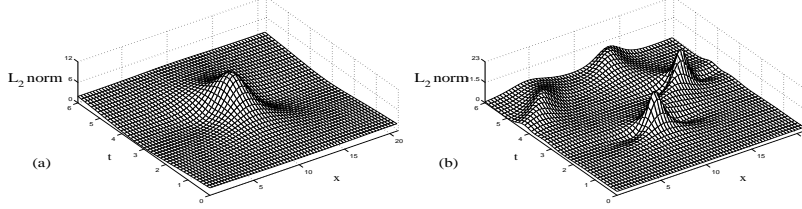


Figure 1. Plot of the L_2 norm versus time and x for the focusing-focusing case ($\sigma_1 = \sigma_2 = 1$) and (a) $\varepsilon = 0$, (b) $\varepsilon = 0.1$ and the following parameters: $\kappa = 0.3$, $a = b = 1$, $a_i = d_i = \nu = \Gamma_i = 1$ (for $i = 1, 2$), $\delta_1 = 0.1$ and $\omega = 1$.

and $\mu \in \mathbb{C}$ satisfies the dispersion relation for linearized disturbances

$$[\mu^2 + (\sigma_1|a|^2 - \kappa^2)] [\mu^2 + (\sigma_2|b|^2 - \kappa^2)] - \sigma_1\sigma_2|a|^2|b|^2 = 0. \quad (16)$$

The dispersion relation (16) can be solved directly for μ yielding four roots

$$\mu_{1/2} = \pm\kappa, \quad \mu_{3/4} = \pm\sqrt{\kappa^2 - \sigma_1 a^2 - \sigma_2 b^2}, \quad (17)$$

with two stable modes ($\mu_{1/2}$) and two potentially unstable modes ($\mu_{3/4}$). A non-zero imaginary part of the phase velocity μ represents a temporal growth mode of instability in time. One can identify a critical wave number κ_{crit} for which $\mu_{3/4}$ is complex, having two instable modes, given by

$$\kappa_{crit} < \sqrt{\sigma_1 a^2 + \sigma_2 b^2}. \quad (18)$$

From equation (18) it is clear that for the focusing-defocusing case ($\sigma_1 = \pm 1, \sigma_2 = \mp 1$) the focusing channel has to be stronger than the defocusing channel for $\mu_{3/4}$ to be complex.

To satisfy the even symmetric boundary conditions (1.2) only the even Fourier cosine modes are chosen and the plane wave number is set to zero ($k = 0$) yielding the initial conditions

$$\begin{aligned} p(x, 0) &= a(1 + (f_+ + f_-^*) \cos(\kappa x + \phi)), \\ q(x, 0) &= b(1 + (g_+ + g_-^*) \cos(\kappa x + \phi)), \end{aligned} \quad (19)$$

where ϕ is an arbitrary phase shift.

Numerical results are shown with and without perturbations for the focusing-focusing case in Figure 1 and the focusing-defocusing case in Figure 2. In both cases homoclinic solutions clearly persist under perturbations, exhibiting pulse shortening and pulse intensification.

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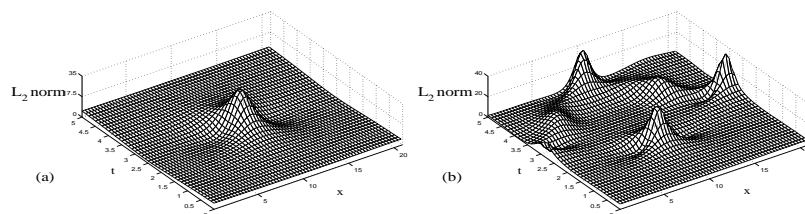


Figure 2. Plot of the L_2 norm versus time and x for the focusing-defocusing case ($\sigma_1 = 1, \sigma_2 = -1$) and (a) $\varepsilon = 0$, (b) $\varepsilon = 0.1$ and the following parameters: $\kappa = 0.3$, $a = 2$, $b = 1$, $a_i = d_i = \nu = \Gamma_i = 1$ (for $i = 1, 2$), $\delta_1 = 0.1$ and $\omega = 1$.

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